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PERTURBATION ANALYSIS OF SHEAR STRAIN LOCALIZATION IN RATE SENSITIVE MATERIALS

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Abstract—The localization of deformation into a shear band is discussed for rate dependent plastic materials. A shear layer model is proposed and the assumption is made that the material considered exhibits strain softening behavior in shear. The results of the shear layer model show that even small material rate sensitivity retards significantly the localization process. A linearized perturbation analysis underestimates the excess strain in the band, but it does provide a qualitative description of strain localization.

1. INTRODUCTION

The localization of deformation into shear bands is commonly observed when solids deform into the plastic range. The localization phenomenon may be understood as a bifurcation from a homogeneous deformation pattern, or, alternatively, it may be considered to result or grow from the material inhomogeneities which would degrade the strength of the material abruptly at the onset of localization. The theoretical framework which governs shear band formation in rate independent solids is due to Hadamard[1], Thomas[2], Hill[3], and Mandel[4]. Rice[5] has given a general review of the theoretical framework for localization of deformation in rate independent materials, both from the bifurcation and the imperfection growth viewpoints. The results of localization analyses depend critically on the constitutive descriptions of the material behavior; for example, see Rice[5].

Plastic deformation via dislocation motion is an inherently rate dependent process. This material rate sensitivity is found to have retarding effects on flow localization. Examples include the study of the necking of a bar by Hutchinson and Neale[6], the study of the necking of a thin sheet by Hutchinson and Neale[7], Marciniak *et al.*[8], and Ghosh[9], and the study of the shear band instability in a porous solid by Pan *et al.*[10].

Here, a general linearized perturbation analysis is presented, and compared with the theoretical framework of the band mode for rate dependent materials; a shear layer model with an assumption of strain softening behavior in shear is proposed to illustrate the retardation effects on flow localization due to the material rate sensitivity.

2. LINEARIZED PERTURBATION ANALYSIS

First, we consider a material element which is rate dependent. For simplicity in this introductory section we assume that the rate dependence is expressed by the following functional relationship:

$$\underset{\sim}{\sigma} = \underset{\sim}{f}(\underset{\sim}{\epsilon}, \dot{\underset{\sim}{\epsilon}}, p_{(1)}, p_{(2)}, \dots p_{(T)}) \tag{2.1}$$

where $\underset{\sim}{\sigma}$ represents the stress tensor and, working within the context of the classical small strain approach, $\underset{\sim}{\epsilon}$ is the (small) strain tensor, $\dot{\underset{\sim}{\epsilon}}$ is the strain rate tensor, and $p_{(1)}, p_{(2)}, \dots$ and $p_{(T)}$ represent properties of the solid. This form oversimplifies what could be regarded as viable models of elastic–plastic rate-sensitive response, but serves to illustrate general effects of rate sensitivity.

The linearized perturbation of eqn (2.1) is

$$\underset{\sim}{\sigma} = \underset{\sim}{f}^0(\underset{\sim}{\epsilon}^0, \dot{\underset{\sim}{\epsilon}}^0, p^0_{(1)}, p^0_{(2)}, \dots, p^0_{(T)}) + \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\epsilon}} : \tilde{\underset{\sim}{\epsilon}} + \frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{\epsilon}}} : \tilde{\dot{\underset{\sim}{\epsilon}}} + \sum_{i=1}^{T} \frac{\partial \underset{\sim}{f}}{\partial p_{(i)}} \tilde{p}_{(i)} \tag{2.2}$$

where the quantities with superscript "0" represent the unperturbed homogeneous strain history

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$\epsilon^0(t)$ and its rate $\dot{\epsilon}^0(t)$ acting on a material with uniform properties $p^0_{(T)}$, a quantity with a tilde "~" represents a small perturbation of the quantity, and the partial derivatives are evaluated at the unperturbed state.

We denote the fourth order tensors $\partial f/\partial \epsilon$ and $\partial f/\partial \dot{\epsilon}$ as L and M, respectively, and we denote the second order tensors $\partial f/\partial p_{(i)}$ as $N^{(i)}$.

The equilibrium condition requires that

$$\nabla \cdot \sigma = 0. \tag{2.3}$$

If f^0 represents the fundamental solution, therefore,

$$\nabla \cdot f^0 = 0.$$

Substituting eqn (2.2) into eqn (2.3), we obtain the equation

$$\nabla \cdot (L(t) : \tilde{\epsilon} + M(t) : \dot{\tilde{\epsilon}} + \sum_{i=1}^{T} N^{(i)}(t)\tilde{p}_{(i)}) = 0. \tag{2.4}$$

The strain and strain rate tensors are

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \tag{2.5}$$

$$\dot{\epsilon} = \frac{1}{2}(\nabla \dot{u} + (\nabla \dot{u})^T) \tag{2.6}$$

where ∇u represents the displacement gradient, $\nabla \dot{u}$ represents the velocity gradient, and a superscript "T" indicates the transpose. We have the perturbed $\tilde{\epsilon}$ and $\dot{\tilde{\epsilon}}$ as function of $\tilde{u}$ and $\dot{\tilde{u}}$, respectively:

$$\tilde{\epsilon} = \frac{1}{2}(\nabla \tilde{u} + (\nabla \tilde{u})^T) \tag{2.7}$$

$$\dot{\tilde{\epsilon}} = \frac{1}{2}(\nabla \dot{\tilde{u}} + (\nabla \dot{\tilde{u}})^T). \tag{2.8}$$

We thus obtain the governing equations for the perturbed $\tilde{u}$ from eqs. (2.4), (2.7) and (2.8). In component form, we have

$$L_{ijkl}(t)\frac{\partial^2 \tilde{u}_k}{\partial x_i \partial x_l} + M_{ijkl}(t)\frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{u}_k}{\partial x_i \partial x_l}\right) + \sum_{l=1}^{T} N^{(l)}_{ij}\frac{\partial \tilde{p}_{(l)}}{\partial x_i} = 0. \tag{2.9}$$

In obtaining eqn (2.9), we use $L_{ijkl} = L_{ijlk}$ and $M_{ijkl} = M_{ijlk}$ because of symmetry of $\tilde{\epsilon}$ and $\dot{\tilde{\epsilon}}$.

The Fourier transformations of $\tilde{u}$ and $\tilde{p}_{(i)}$ are

$$U(k, t) = \int_x \tilde{u}(x, t)\, e^{-\sqrt{-1}k \cdot x}\, d^3x$$

$$P_{(i)}(k, t) = \int_x p_{(i)}(x, t)\, e^{-\sqrt{-1}k \cdot x}\, d^3x.$$

Equation (2.9) becomes

$$k_i L_{ijml}(t) k_l U_m(k, t) + k_i M_{ijml}(t) k_l \frac{\partial}{\partial t} U_m(k, t) - \sum_{l=1}^{T} \sqrt{-1}\, k_i N^{(l)}_{ij}(t) P_{(l)} = 0. \tag{2.10}$$

Now, we consider the special case of a band mode where $\tilde{u}$ and $\tilde{p}_{(i)}$ only vary along the direction of n, so that

$$\tilde{u} = \tilde{u}(n \cdot x, t)$$

$$\tilde{p}_{(i)} = \tilde{p}_{(i)}(n \cdot x, t),$$

Equation (2.9) becomes

$$n_i L_{ijkl}(t) n_l \tilde{u}''_k(n \cdot x, t) + n_i M_{ijkl}(t) n_l \frac{\partial}{\partial t} \tilde{u}''_k(n \cdot x, t) + \sum_{l=1}^{T} n_i N_{ij}^{(l)}(t) \tilde{p}'_{(l)}(n \cdot x, t) = 0 \qquad (2.11)$$

where $\tilde{u}''_k(n \cdot x, t)$ means the second derivative of $\tilde{u}_k(n \cdot x, t)$ with respect to $n \cdot x$, and $\tilde{p}'_{(l)}(n \cdot x, t)$ means the first derivative of $\tilde{p}_{(l)}(n \cdot x, t)$ with respect to $n \cdot x$.

When the material is rate independent, we have

$$\frac{\partial f}{\partial \dot{\epsilon}} = M \equiv 0$$

so the second term, which has the time derivative in eqns (2.10) and (2.11), vanishes. The localization condition for the band mode (eqn (2.11)) is that $\det(n \cdot L \cdot n) = 0$ (where $\det(X)$ means the determinant of the matrix X); this is the bifurcation condition on the small strain analysis. From a number of analyses of specific materials with the presence of an initial inhomogeneity the localization condition $\det(n \cdot L \cdot n) = 0$ in the imperfection band is met well before the moduli L of the outer field would allow localization, as in Rice[5]. The linearized perturbation analysis, vis-a-vis the inhomogeneity formulation, thus overestimates the material strength. However, the linearized perturbation analysis does give the correct bifurcation condition to a band mode from a homogeneous deformation. When $L_{ijkl} = L_{klij}$, the loss of uniqueness of a material element subjected to overall displacement boundary conditions occurs as soon as the condition $\det(k \cdot L \cdot k) = 0$ becomes possible for some k in k-space; see Rice[5]. This links localization into the band mode to the uniqueness of the boundary value problem. For the band mode, we thus define "material instability" as occuring when the condition $\det(n \cdot L \cdot n) = 0$ for some oreintation n is first met during the deformation program.

For the rate dependent material, we similarly can treat the "material instability" as the possibility that some component of $\tilde{u}(n \cdot x, t)$ of the band mode may grow without limit for some orientation n.

For a concise general analysis of the rate dependent case, rewrite eqn (2.11) as

$$(n \cdot M \cdot n) \cdot \dot{\tilde{u}}'' + (n \cdot L \cdot n) \cdot \tilde{u}'' = q \qquad (2.12)$$

where $q = q(t)$ may here be regarded as given, well-behaved functions of time, representing the last terms in eqn (2.11). Observe that $n \cdot M \cdot n$ and $n \cdot L \cdot n$ may be regarded, for any given n, as 3×3 matrices. If we limit attention to materials with "positive" instantaneous viscosity, it is appropriate to assume that $n \cdot M \cdot n$ is a positive definite matrix, and hence that its three eigenvalues m_1, m_2, m_3 have positive real part and, further, that $\det(n \cdot M \cdot n) = m_1 m_2 m_3 > 0$. On the other hand, $n \cdot L \cdot n$ is positive definite only up to the point ϵ^{0*} in the strain history at which rate independent localization condition is met, namely up to the point at which $\det(n \cdot L \cdot n) = 0$. Prior to this point we may assume that all the three eigenvalues l_1, l_2, l_3 of $n \cdot L \cdot n$ have positive real part, whereas at the critical point the real part of the least eigenvalue passes through zero and, subsequently, has negative real part. (Of course, in the particular case when $n \cdot L \cdot n$ is a symmetric matrix, i.e. when $L_{ijkl} = L_{klij}$, the l_1, l_2, l_3 are always real.)

Accordingly, the matrix inverse of $n \cdot M \cdot n$ exists and one may write

$$\dot{\tilde{u}}'' + (n \cdot M \cdot n)^{-1} \cdot (n \cdot L \cdot n) \cdot \tilde{u}'' = h \qquad (2.13)$$

where $h = (n \cdot M \cdot n)^{-1} \cdot q$. It is well known that this equation exhibits stable solutions (no exponential growth), so long as the eigenvalues a_1, a_2, a_3 of the 3×3 matrix A, where

$$\underset{\sim}{A} \equiv (\underset{\sim}{n}\cdot\underset{\sim}{M}\cdot\underset{\sim}{n})^{-1}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{L}\cdot\underset{\sim}{n}),$$

have positive real part. On the other hand, if any of the eigenvalues a_1, a_2, a_3 of $\underset{\sim}{A}$ have negative real part, the solutions for $\underset{\sim}{\tilde{u}}''$ are unstable in the sense of exhibiting exponential growth in time. Now, since

$$\det(\underset{\sim}{A}) = \det(\underset{\sim}{n}\cdot\underset{\sim}{L}\cdot\underset{\sim}{n})/\det(\underset{\sim}{n}\cdot\underset{\sim}{M}\cdot\underset{\sim}{n}),$$

it follows that

$$a_1a_2a_3 = l_1l_2l_3/m_1m_2m_3.$$

This result shows, since $m_1m_2m_3 > 0$, that at the critical condition for localization according to the rate independent criterion, namely at the state when one of the l's passes through zero, one of the a's must pass through zero also. Hence, when the rate independent criterion for localization is met, the solutions to the rate dependent model change character from exponential decay (Re $a > 0$) to exponential growth (Re $a < 0$). Nevertheless, as the detailed calculations to follow verify, this growth procedes at a finite rate and, in the presence of non-negligible rate-sensitivity, the onset of growth (i.e. the meeting of the rate independent localization condition) cannot be identified sensibly with instability.

3. LINEAR STABILITY ANALYSIS OF A SIMPLE SHEAR LAYER MODEL

A narrow band of intense shear deformation is a commonly observed phenomenon in experiments; also, in single crystals or polycrystalline aggregates, the plastic deformation is generally considered as shear deformation on some specific crystallographic planes of each grain via crystalline slip or dislocation motion. Thus, we are led to investigate the material behavior under shear deformation.

We consider an infinite layer with the normal $\underset{\sim}{n}$ in the y direction as shown in Fig. 1. A shear traction σ_{xy} is applied on the layer. The properties of the layer are uniform in the x and z directions, but are allowed to have a small perturbation in the y direction. All of the field quantities vanish, except u_x, the displacement in the x direction, γ_{xy}, the engineering shear strain $\partial u_x/\partial y$ and σ_{xy}, the shear stress.

The equilibrium condition is that

$$\frac{\partial \sigma_{xy}}{\partial y} = 0. \tag{3.1}$$

This requires that σ_{xy} is uniform in the y direction, and is a function of time t only. Thus, we have

$$\begin{aligned} \tilde{\sigma}_{xy} &= 0 \\ \dot{\tilde{\sigma}}_{xy} &= 0 \end{aligned} \tag{3.2}$$

where a quantity with a dot means its time derivative.

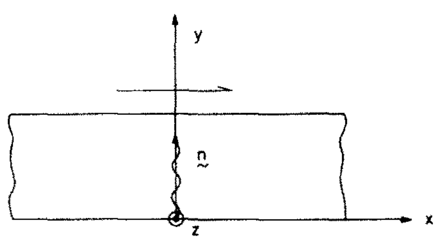

Fig. 1. An infinite layer with a small perturbation in the y direction subjected to a shear stress σ_{xy}.

Now, we consider the layer as elastic–viscoplastic, and decompose the shear strain rate $\dot{\gamma}_{xy}$ into the elastic part $\dot{\gamma}^e_{xy}$ and the plastic part $\dot{\gamma}^p_{xy}$.

$$\dot{\gamma}_{xy} = \dot{\gamma}^e_{xy} + \dot{\gamma}^p_{xy}. \tag{3.3}$$

The elastic part $\dot{\gamma}^e_{xy}$ is related to $\dot{\sigma}_{xy}$ by the shear modulus G as

$$\dot{\gamma}^e_{xy} = \frac{\dot{\sigma}_{xy}}{G}. \tag{3.4}$$

A power rate hardening law is used to describe the material rate dependent behavior in shear

$$\dot{\gamma}^p_{xy} = \dot{a}\left(\frac{\sigma_{xy}}{g(\gamma^p_{xy})}\right)^{1/m} \tag{3.5}$$

where $\dot{a}$ is regarded as the reference plastic shear strain rate, $g(\gamma^p_{xy})$ represents the shear stress σ_{xy} when $\dot{\gamma}^p_{xy}$ is set equal to $\dot{a}$, and m is the plastic strain rate sensitivity, defined as

$$m = \frac{\partial ln\sigma_{xy}}{\partial ln\dot{\gamma}^p_{xy}}. \tag{3.6}$$

The perturbation of the elastic strain rate $\dot{\gamma}^e_{xy}$ is

$$\dot{\tilde{\gamma}}^e_{xy} = \frac{\dot{\tilde{\sigma}}_{xy}}{G} = 0. \tag{3.7}$$

Here, the elastic modulus G is treated as a uniform property throughout the layer so that the plastic behaviour of the layer can be investigated.

The linearized perturbation of eqn (3.5) is

$$\dot{\tilde{\gamma}}^p_{xy} = \frac{\dot{\gamma}^p_{xy}}{m\sigma_{xy}}\left(\tilde{T} + \tilde{\sigma}_{xy} - \sigma_{xy}\frac{g'(\gamma^p_{xy})}{g(\gamma^p_{xy})}\tilde{\gamma}^p_{xy}\right) \tag{3.8}$$

where $g'(\gamma^p_{xy})$ represents the derivative of $g(\gamma^p_{xy})$ with respect to γ^p_{xy}, and $\tilde{T}$ represents the perturbation of the material properties

$$\tilde{T} = m\sigma_{xy}\left[\frac{\tilde{\dot{a}}}{\dot{a}} - \frac{1}{m}\frac{\tilde{g}(\gamma^p_{xy})}{g(\gamma^p_{xy})} - \frac{\tilde{m}}{m^2}\ln\left(\frac{\sigma_{xy}}{g(\gamma^p_{xy})}\right)\right]. \tag{3.9}$$

Since $\tilde{\sigma}_{xy} = 0$, eqn (3.8) can be rearranged as

$$\frac{d\tilde{\gamma}^p_{xy}}{d\gamma^p_{xy}} + \frac{1}{m}\frac{g'(\gamma^p_{xy})}{g(\gamma^p_{xy})}\tilde{\gamma}^p_{xy} = \frac{1}{m\sigma_{xy}}\tilde{T}. \tag{3.10}$$

We rewrite eqn (3.10) as

$$\frac{d\tilde{\gamma}}{d\gamma} + \frac{g'(\gamma)}{mg(\gamma)}\tilde{\gamma} = \frac{\tilde{T}(\gamma)}{m\tau(\gamma)} \tag{3.11}$$

where γ replaces γ^p_{xy}, and τ replaces σ_{xy}. This is an inhomogeneous first-order ordinary differential equation (with respect to γ). The general solution is

$$\tilde{\gamma}(\gamma) = \int_0^\gamma \frac{\tilde{T}(\gamma')}{m\tau(\gamma')}\left(\frac{g(\gamma')}{g(\gamma)}\right)^{1/m} d\gamma' + c\left(\frac{g(0)}{g(\gamma)}\right)^{1/m}. \tag{3.12}$$

The first term of the solution is the particular solution, and represents the contribution to $\tilde{\gamma}$

from the perturbation of the material properties; the second term represents the complementary solution, and the constant c is identified as $\tilde{\gamma}(0)$ which suggests that this part of the solution deals with the development of the initial perturbed shear strain $\tilde{\gamma}(0)$.

We assume $\dot{\gamma} = \dot{a}$ so that $g(\gamma) = \tau$. The complementary solution equals $\tilde{\gamma}(0)\ (g(0)/g(\gamma))^{1/m}$, and is larger than $\tilde{\gamma}(0)$ only when $g(\gamma) < g(0)$; the complementary solution does not grow until the shear stress is less than the shear stress at zero plastic strain.

The material may become strain softening due to various physical mechanisms, such as heat from a large amount of plastic work[11], or void nucleation and growth[10]. Here, we assume strain softening behavior in shear without investigating its physical basis. We denote the maximum shear stress and its corresponding plastic strain as τ_p and γ_p, respectively. Furthermore, we denote $\tilde{T}(\gamma_p)$ as $\tilde{T}$. By assuming m is a small constant, we can apply Laplace's method and obtain asymptotic results when $\tilde{\gamma}$ is near the peak of τ for the particular solution. The results are:

$$\tilde{\gamma} = m^{-1/2}(1-m)^{-1/2}\frac{\tilde{T}}{\tau(\gamma)}\left(\frac{\tau_p}{\tau(\gamma)}\right)^{(1-m)/m}\left(\frac{2\tau_p}{\kappa}\right)^{1/2}\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\left(\frac{1-m}{m}\right)^{1/2}\left(\frac{\kappa}{2\tau_p}\right)^{1/2}(\gamma_p-\gamma)\right) \tag{3.13}$$

when $\gamma < \gamma_p$

$$\tilde{\gamma} = m^{-1/2}(1-m)^{-1/2}\frac{\tilde{T}}{\tau_p}\left(\frac{2\tau_p}{\kappa}\right)^{1/2}\frac{\sqrt{\pi}}{2} \tag{3.14}$$

when $\gamma = \gamma_p$

$$\tilde{\gamma} = m^{-1/2}(1-m)^{-1/2}\frac{\tilde{T}}{\tau(\gamma)}\left(\frac{\tau_p}{\tau(\gamma)}\right)^{(1-m)m}\left(\frac{2\tau_p}{\kappa}\right)^{1/2}\frac{\sqrt{\pi}}{2}\left(2\text{-erfc}\left(\left(\frac{1-m}{m}\right)^{1/2}\left(\frac{\kappa}{2\tau_p}\right)^{1/2}(\gamma-\gamma_p)\right)\right) \tag{3.15}$$

when $\gamma > \gamma_p$. Here, κ is the absolute curvature at the peak stress τ_p ($\kappa = |g''(\gamma)|$), and erfc(x) represents the complementary error function which quickly decays to zero as x becomes large. Since $\tilde{\gamma}$ is a continuous function around the peak stress τ_p, we can examine eqn (3.14) to get some qualitative descriptions of the growing processes; here, $\tilde{\gamma}$ is proportional to the material perturbation $\tilde{T}$, and has square root singularities in both the rate sensitivity m and the curvature κ. As m becomes smaller to reach the limit of rate independent behavior, the layer becomes more unstable near the neighborhood of the peak stress; as the stress–strain curve becomes flatter near the peak stress, the perturbation of γ grows in the layer. This is consistent with a corresponding linearized rate independent analysis: if we assume that the layer is rate independent, and its stress–strain relation is $\tau = g(\gamma)$, the linearized perturbation of the governing equation becomes

$$g'(\gamma)\tilde{\gamma} + \tilde{g}(\gamma) = 0. \tag{3.16}$$

Thus, $\tilde{\gamma}$ becomes infinite when $g'(\gamma) = 0$ and $\tilde{g}(\gamma) \neq 0$, and may be nonzero when $g'(\gamma) = 0$ and $\tilde{g}(\gamma) = 0$. This is consistent with the bifurcation analysis of Rudnicki and Rice[12] for pure shear. In summary, the above linearized analysis of a layer under shear shows that the material rate sensitivity retards the instability predicted by the corresponding rate independent analysis.

4. NON-LINEAR STABILITY ANALYSIS AND NUMERICAL RESULTS OF THE SIMPLE SHEAR MODEL

Next, we want to examine the validity of the linearized perturbation analysis. We suppose an imperfection in the form of a thin planar band having its normal in the y direction. The band lies within the layer of unit thickness (Fig. 2), has width w, and a different $g(\gamma)$ from the surrounding material. We denote quantities in the band by a subscript "1", and quantities outside the band by a subscript "0". We denote the overall plastic shear strain and its rate as γ and $\dot{\gamma}$, respectively. The stress history is chosen such that $\dot{\gamma} = \dot{a} =$ constant, and, therefore,

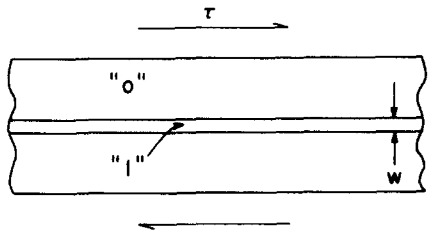

Fig. 2. An imperfection in the form of a thin planar band of width w lying within a layer of unit thickness.

$\tau = g(\gamma) = g(\dot{a}t)$. Then eqn (3.5) can be rewritten for both region "0" and region "1" as

$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}\gamma} = \left(\frac{\tau}{g_0(\gamma_0)}\right)^{1/m} \tag{4.1}$$

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}\gamma} = \left(\frac{\tau}{g_1(\gamma_1)}\right)^{1/m} \tag{4.2}$$

since the equilibrium condition requires τ to be homogeneous in the y direction. The overall increment $\mathrm{d}\gamma$ is

$$\mathrm{d}\gamma = w\,\mathrm{d}\gamma_1 + (1-w)\,\mathrm{d}\gamma_0. \tag{4.3}$$

The stress τ can be obtained from eqns (4.1)–(4.3) as

$$\tau = \{w(g_1(\gamma_1))^{-1/m} + (1-w)(g_0(\gamma_0))^{-1/m}\}^{-m}. \tag{4.4}$$

The elastic shear strains of both regions are equal to τ/G.

We assume a quadratic form for $g(\gamma)$ to approximate the strain softening behavior in the neighborhood of γ_p as

$$g(\gamma) = \tau_p - \frac{1}{2}\kappa(\gamma-\gamma_p)^2 \tag{4.5}$$

where κ is the curvature at the peak stress, and τ_p and γ_p are the maximum stress and the corresponding strain.

In region "1", we assume a slight perturbation in the property $g(\gamma)$ as

$$g_1(\gamma_1) = g(\gamma_1) + \tilde{g}(\gamma_1) \tag{4.6}$$

where $\tilde{g}(\gamma_1)$ is presently assumed to be constant.

In region "0", we do not perturb the function $g(\gamma)$ and we have

$$g_0(\gamma_0) = g(\gamma_0). \tag{4.7}$$

The width w of the band generally relates to the microstructure of the material. The analysis here has no significant length scale which can give an estimate of the width w. We assume that the band has infinitesimally small width $w = 0$ so that the effects on localization of deformation due to rate sensitivities and initial inhomogeneities can be more readily investigated.

In Fig. 3, we plot the exact results for $\tilde{T} = (-\tilde{g}) = 0.01\ \tau_p$, $\gamma_p = 0.15$, and $\tau_0/\tau_p = 0.9$, where τ_0 is the shear stress at zero plastic shear strain. The rescaled shear strain difference of both regions, $(\kappa/2\tau_p)^{1/2}\tau(\gamma_0-\gamma_1)/\tilde{T}$, are plotted as a function of $(\kappa/2\tau_p)^{1/2}(\gamma_0-\gamma_p)$. The curve marked $m=0$ represents the corresponding rate independent solution. For the rate independent analysis, the function g is now the stress function. The equilibrium condition requires

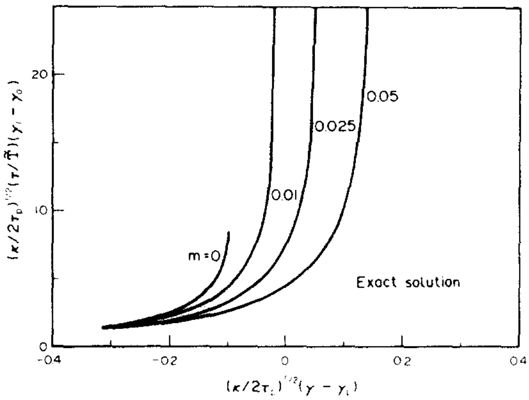

Fig. 3. Comparison of the growth of the excess shear strain in the band for different m.

$g_0(\gamma_0) = g_1(\gamma_1)$. The maximum strain $\bar{\gamma}_0^{cr}$ which can be attained in the region "0" is

$$\bar{\gamma}_0^{cr} = \gamma_p\left(1 - \left(\frac{\tilde{T}}{\tau_p(1 - \tau_0/\tau_p)}\right)^{1/2}\right). \tag{4.8}$$

There is no valid solution for region "0" after $\gamma = \bar{\gamma}_0^{cr}$ unless the layer unloads. For the rate dependent case ($m \neq 0$), the curves are simply obtained by integrating eqns (4.1), (4.2) and (4.4) incrementally. We can see that the shear strain in the band grows slowly at first, but that an extremely fast growth in shear strain in the band eventually takes over. Here, the localization strain γ_0^{cr} is defined at the moment when $d\gamma_1/d\gamma_0 = \infty$ is reached; the localization condition is met when γ_1 reaches a value such that $g(\gamma_1) = 0$ because the quadratic form of $g(\gamma_1)$ is assumed.

Since we assume that the band width w is infinitesimally small ($w = 0$), we have the overall plastic strain γ equivalent to γ_0. If we assume a finite band width, according to the rate dependent analysis, the overall shear plastic strain is infinite when $d\gamma_1/d\gamma_0 = \infty$ (see eqn 4.3).

We define $\delta\gamma$ as

$$\delta\gamma = \gamma_0^{cr} - \bar{\gamma}_0^{cr}.$$

In Fig. 4, the quantity $(\kappa/2\ \tau_p)^{1/2}\ \delta\gamma$ is plotted against m. We can see that the retardation effect on strain localization is larger when m is larger and when the inhomogeneity $\tilde{T}$ is smaller. From the numberical solutions, we can write approximately

$$\delta\gamma = \gamma_p f(\tilde{T}) m^{0.675}/(1 - \tau_0/\tau_p)^{1/2} \tag{4.9}$$

where $f(\tilde{T})$ is a constant which depends on the imperfection level $\tilde{T}$. The detailed asymptotic analysis of the retardation on localization for very small m is given in the appendix.

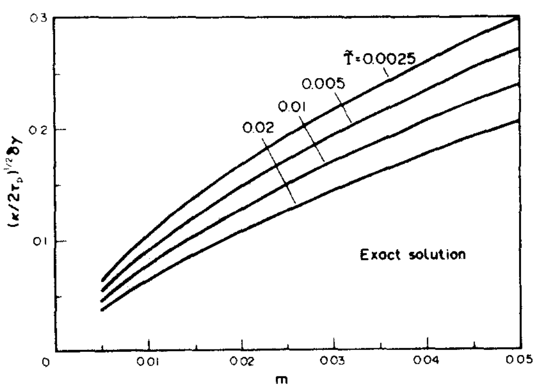

Fig. 4. The normalized additional maximum strain plotted as a function of m for different imperfection level $\tilde{T}$.

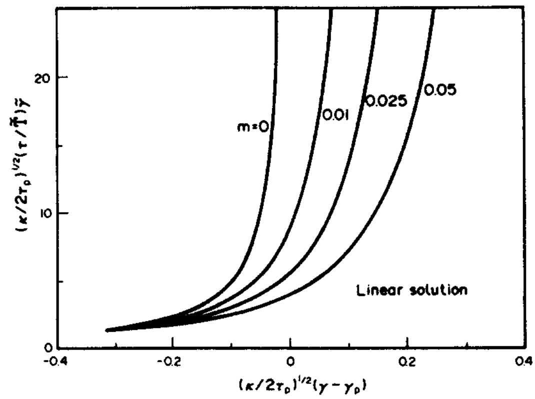

Fig. 5. Comparison of the growth of the perturbed strain for different m.

Figure 5 shows the numerical results of the linearized perturbation analysis for $\gamma_p = 0.15$, $\bar{T} = 0.01$, and $\tau_0/\tau_p = 0.9$. By comparing this to Fig. 3, we can see that the linearized analysis significantly underestimates the excess strain in the band. However, it can show the trend of the effects on localization due to the rate sensitivity.

In Fig. 6, solutions to the linearized perturbation scheme both from the asymptotic analysis and from the straightforward incremental integration of eqn (3.11) are shown for $\gamma_p = 0.15$, $\bar{T} = 0.01$, and $\tau_0/\tau_p = 0.9$. In the plot, the dash lines represent the asymptotic solutions and the solid lines represent the direct integrations of eqn (3.11). The plots show good agreement in predictions of the perturbed shear strain $\tilde{\gamma}$ and that, as m gets smaller, the prediction of the solution by using the asymptotic result is better.

In Fig. 7, a comparison of various analysis for $\gamma_p = 0.15$, $\bar{T} = 0.01$, and $\tau_0/\tau_p = 0.9$ is plotted. Curve 1 represents the rate independent ($m = 0$) exact solution; curve 2 is the linear perturbation results of the rate independent ($m = 0$) analysis; curve 3 is the rate dependent ($m = 0.025$) exact solution which gives a larger localization strain; curve 5 is the linear perturbation results of the rate dependent ($m = 0.025$) analysis which underestimates the perturbed strain $\tilde{\gamma}$ in the band. We note that the localization strain of curve 2 may move toward the bifurcation shear strain γ_p as $\bar{T}$ is smaller, and that the localization strain of curve 3 may be less than the bifurcation shear strain γ_p when m is smaller or $\bar{T}$ is larger. The governing equation of the rate dependent perturbation analysis with the second order correction is

$$\frac{d\tilde{\gamma}}{d\gamma} = \left(-\tilde{g}(\gamma) - g'(\gamma)\tilde{\gamma} - \frac{1}{2}g''(\gamma)\tilde{\gamma}^2\right)/mg(\gamma). \tag{4.10}$$

Here, $g(\gamma)$ is assumed to be quadratic in γ so that $g'''(\gamma) = 0$; therefore, eqn (4.10) is the exact

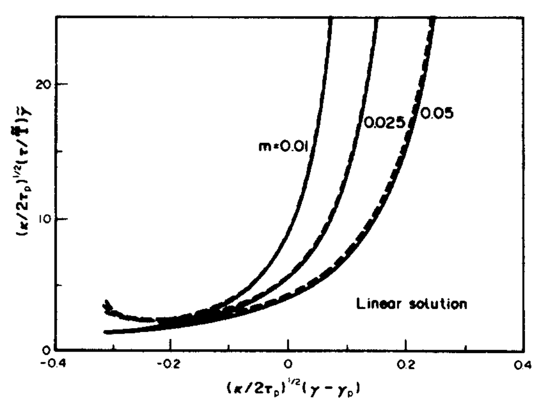

Fig. 6. Comparison of the solutions to the linearized perturbation scheme. The dashed lines represent the asymptotic solutions and the solid lines represent the straightforward integration of eqn (3.11).

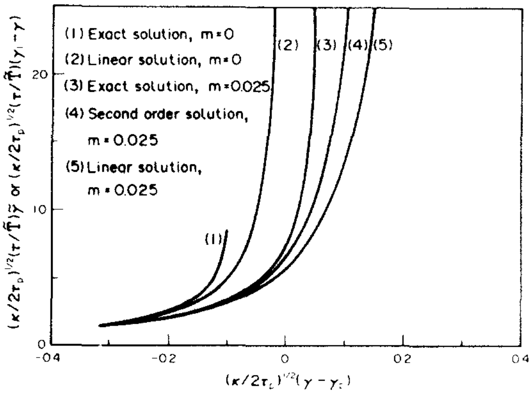

Fig. 7. Comparison of the solutions for various analyses.

expansion. If the term which contains $g''(\gamma)$ is dropped, eqn (4.10) is reduced to eqn (3.11) because of $\tilde{T} = -\tilde{g}(\gamma)$ and $\tau(\gamma) = g(\gamma)$. Curve 4 is the direct numerical integration of the above equation ($m = 0.025$). The results are better than the linearized analysis, but, the excess strain in the band is still underestimated.

Now, we choose the power rate hardening law as

$$\tau = g(\gamma)\dot{\gamma}^m \tag{4.11}$$

and, particularly, assume

$$g(\gamma) = k\,e^{-\gamma}\gamma^N \tag{4.12}$$

where k and N are constants. Equation (4.12) gives strain softening behavior. For the layer, we assume

$$g_0(\gamma_0) = k\,e^{\gamma_0}\gamma_0^N \tag{4.13}$$

$$g_1(\gamma_1) = (1-\eta)k\,e^{\gamma_1}\gamma_1^N \tag{4.14}$$

where η represents the imperfection in the band.

For the rate dependent analysis, the equilibrium condition requires

$$k\,e^{\gamma_0}\gamma_o^N\dot{\gamma}_0^m = (1-\eta)k\,e^{-\gamma_1}\gamma_1^N\dot{\gamma}_1^m \tag{4.15}$$

For the corresponding rate independent analysis, the equilibrium condition requires

$$k\,e^{-\gamma_0}\gamma_0^N = (1-\eta)k\,e^{-\gamma_1}\gamma_1^N \tag{4.16}$$

The governing equations (4.15) and (4.16) are exactly the same as those derived by Hutchinson and Neale[6] for a necked tensile bar. Their results may be summarized as follows: When rate sensitivity is not considered, the maximum shear strain $\bar{\gamma}_0^{cr}$ occurs when $\gamma_1 = N$ and is given by the relation

$$e^{-N}N^N = \frac{1}{1-\eta}e^{-\bar{\gamma}_0^{cr}}(\bar{\gamma}_0^{cr})^N \tag{4.17}$$

we note that, when $\eta = 0$, $\bar{\gamma}_0^{cr} = N$ which is the bifurcation shear strain of the layer (or the Considère strain of the tensile bar). For small imperfection η, eqn (4.17) gives

$$\frac{\bar{\gamma}_0^{cr}}{N} \approx 1 - \sqrt{\frac{2\eta}{N}}. \tag{4.18}$$

Equation (4.18) shows that a larger η promotes earlier localization. When rate sensitivity is considered, the localization condition is defined at the instant when γ_1 becomes infinite for the case considered herein. Thus, let $s = 1/m$, and $p = N/m$, so the governing equation (4.15) becomes

$$\int_0^\infty e^{-s\gamma_1}\gamma_1{}^p\, d\gamma_1 = \int_0^{\gamma_0{}^{cr}} \frac{1}{(1-\eta)^s} e^{-s\gamma_0}\gamma_0{}^p\, d\gamma_0. \tag{4.19}$$

For small m, small η and $m \le 2\eta$, the asymptotic solution of eqn (4.19) is

$$\frac{\delta\gamma}{\sqrt{N}} \approx \frac{m}{2\sqrt{2\eta}} \ln\left(\frac{4\pi\eta}{m}\right) \tag{4.20}$$

where $\delta\gamma = \gamma_0^{cr} - \bar{\gamma}_0{}^{cr}$. The retarding effects on localization due to m are clearly seen.

It is very interesting to note that the geometric necking mechanism of the tensile bar, in some sense, is equivalent to some material degrading mechanism in the shear layer model. The localization analysis for both problems is so closely related.

5. CONCLUSION

A general framework for linearized perturbation analysis in rate dependent materials has been established. Specifically, the connection with localization into a band mode has been made.

Without getting involved complex constitutive description of material rate dependent behavior, we have simply investigated the basic shear mechanism of the plastic deformation. By assuming strain softening behavior in shear, we have examined the validity of the linearized analysis, and have found that a fully nonlinear analysis is needed to more accurately describe the localization process. However, the qualitative tendency to localize can be obtained from the linearized perturbation analysis. Also, the retarding effects on the localization of deformation due to the material rate sensitivity has been noted.

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APPENDIX

The governing equations of the shear layer which is considered in the main text are

$$\frac{d\gamma_1}{d\gamma} = \left(\frac{\tau}{g_1(\gamma_1)}\right)^{1/m} \tag{A1}$$

$$\frac{d\gamma_0}{d\gamma} = \left(\frac{\tau}{g_0(\gamma_0)}\right)^{1/m} \tag{A2}$$

When the band width w is assumed to be infinitesimally small, $d\gamma_0 = d\gamma_1$, and $\tau = g_0(\gamma_0)$. If we assume that $g_1(\gamma_1)$ goes to zero fast enough as γ_1 goes infinite, we then define that localization occurs at a critical value of γ_0^{cr} when γ_1 becomes infinite. From eqns (A1) and (A2),

$$\int_0^{\infty} g_1(\gamma_1)^{1/m}\, d\gamma_1 = \int_0^{\gamma_0^{cr}} g_0(\gamma_0)^{1/m}\, d\gamma_0 \tag{A3}$$

when m is a small constant and $g_1(\gamma_1)$ drops to zero fast enough, the major contribution to the integral on the left-hand side of eqn (A3) comes near the peak of $g_1(\gamma_1)$. Applying Laplace's method, we find

$$\int_0^{\gamma_0^{cr}} g_1(\gamma_1)^{1/m}\, d\gamma_1 = (\tau_p - \bar{T})^{1/m}\sqrt{2\pi m}\left(\frac{\tau_p - \bar{T}}{\kappa}\right)^{1/2}. \tag{A4}$$

A very small m is assumed to ensure $\gamma_0^{cr} < \gamma_p$, so that the major contribution to the r.h.s. of eqn (A3) comes near γ_0^{cr}. Applying Laplace's method, we have

$$\int_0^{\gamma_0^{cr}} g_0(\gamma_0)^{1/m}\, d\gamma_0 = m\left(\tau_p - \frac{\kappa}{2}(\gamma_0^{cr} - \gamma_p)^2\right)^{1/m} \frac{\tau_p - \frac{\kappa}{2}(\gamma_0^{cr} - \gamma_p)^2}{-\kappa(\gamma_0^{cr} - \gamma_p)} \tag{A5}$$

Denote

$$\delta\gamma = \gamma_0^{cr} - \bar{\gamma}_0^{cr} \tag{A6}$$

where $\bar{\gamma}_0^{cr} = \gamma_p - (2\bar{T}/\kappa)^{1/2}$ represents the maximum strain that can be attained outside the band for the corresponding rate independent analysis. We can obtain the expression for the small $\delta\gamma$ from eqns (A4)–(A6) as

$$\delta\gamma = \frac{1 - \left(\dfrac{m(\tau_p - \bar{T})\bar{T}}{\pi\kappa^2}\right)^{m/2}}{\dfrac{(2\kappa\bar{T})^{1/2}}{\tau_p - \bar{T}}\left(1 + m\left(\dfrac{m(\tau_p - \bar{T})\bar{T}}{\pi\kappa^2}\right)^{m/2}\dfrac{\tau_p + \bar{T}}{2\bar{T}}\right)}. \tag{A7}$$

The retardation effects on localization due to the rate sensitivity m cannot be readily understood from the complex relationship (A7). In Fig. 8, we plot $(\kappa/2\tau_p)^{1/2}\delta\gamma$ against m according to eqn (A7) for very small values of m. When the imperfection level $\bar{T}$ is varied, the plot shows the same trends as that observed in Fig. 4. It is well known that the numerical integration is very difficult since the differential equation is very stiff when the rate sensitivity m is small; therefore, in Fig. 8, the asymptotic results are plotted for very small m and a good qualitative agreement with the numerical results of the exact solution is shown.

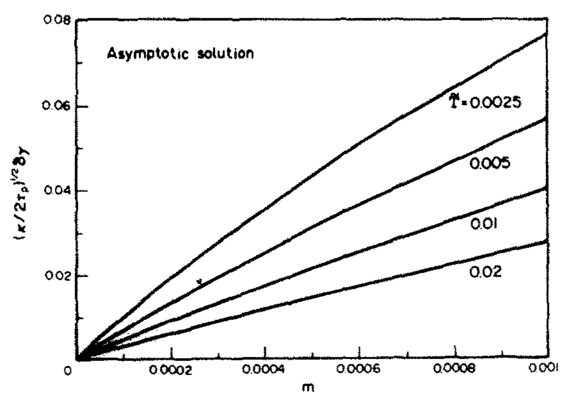

Fig. 8. The asymptotic solutions for the normalized additional maximum strain plotted as functions of m at very small m for different imperfection level $\bar{T}$.